Improved Estimator of Population Mean Using Auxiliary Information under Ranked Set Sampling

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Abstract

A ratio type estimator of population mean has been proposed by using auxiliary information efficiently under ranked set sampling (RSS). The expressions for bias and mean square error (MSE) of the estimator have been obtained. The proposed estimator has been compared with existing estimators of population mean. It has been shown that the proposed estimator is more efficient than the estimators suggested by Samawi and Muttlak (1996), Bouza (2008) and Jemain et al. (2008). The results of the proposed estimator have been verified through simulation study.

Keywords: Ratio estimator, Efficient auxiliary information, Ranked set sampling and Simulation study.

1. INTRODUCTION

RSS has been proved as a cost effective sampling technique which can be used when cost of observing study variable is very high but ranking of units is possible visually or by any other inexpensive method. McIntyre (1952) investigated that mean per unit estimator under RSS is more efficient as compared to the simple random sampling (SRS). The necessary mathematical theory of RSS was provided by Takahasi and Wakimoto(1968) . Samawi and Muttlak (1996) suggested a ratio estimator for population ratio, which can also be used to estimate population mean or population total. It has been found that the classical ratio estimator is more efficient under RSS than under SRS. Performance of classical product estimator is better under RSS as compared to under SRS [see Bouza (2008)]. When population correlation coefficient is known, Jemain et al. (2008) proposed an estimator under RSS and they proved that the proposed estimator is more efficient than the estimator proposed by Singh and Tailor (2003).

Sahai and Sahai (1985) developed new ratio-type estimator by using auxiliary information efficiently under SRS. In the present paper, a ratio-type estimator of population
mean has also been proposed by using auxiliary information efficiently under RSS. The expressions for bias and mean square error of the estimator have been obtained. Also, it has been proved that performance of proposed estimator is better than the estimators suggested by Samawi and Muttlak (1996), Bouza (2008) and Jemain et al. (2008).

1.1 Method of selecting a Ranked Set Sample

The RSS procedure for selecting a sample of size \( n' \), involves selecting randomly \( n' \) units from the population and allocating randomly into \( n' \) equal samples. The \( n' \) units of each sample are ranked visually or by any inexpensive method with respect to the study variable. From the first set of \( n' \) units, the smallest unit is taken and from the second set, the second smallest of \( n' \) units is taken. The process is continued until up to the \( n'th \) set, from where the largest of the \( n' \) units is measured. Repeating the process \( m \) times yields a sample of size \( mn' = n \) from \( mn'^2 \) units. Due to \( m \) repetitions, the \( MSE \) of the estimators decreases \( m \) times.

1.2 Under SRS

Let a simple random sample \((y_1, x_1), (y_2, x_2), ..., (y_n, x_n)\) of size \( n \) is drawn from the population of size \( N \). Sahai and Sahai (1985) suggested an estimator of population as given below

$$
\bar{y}_{SS} = \frac{y(1+\alpha)\bar{X} + (1-\alpha)\bar{x}}{(1+\alpha)\bar{x} + (1-\alpha)\bar{X}}
$$

(1)

where \( \bar{y} \) and \( \bar{x} \) denote the sample means of study variable and auxiliary variable respectively, \( \bar{Y} \) and \( \bar{X} \) denote the population means of study variable and auxiliary variable respectively and \( \alpha \) is a pre-assigned constant.

The approximate expressions for bias and MSE of \( \bar{y}_{SS} \) are as follows:

$$
\text{Bias}(\bar{y}_{SS}) = \frac{1}{2} \frac{\text{Var}(\bar{x})}{\bar{X}^2} - \alpha \frac{\text{Cov}(\bar{x}, \bar{y})}{\bar{X}\bar{Y}}
$$

(2)

and

$$
\text{MSE}(\bar{y}_{SS}) = \text{Var}(\bar{y}) + \alpha^2 \text{Var}(\bar{x}) - 2\alpha \text{RCov}(\bar{x}, \bar{y})
$$

(3)

where \( \text{Var}(\bar{y}) = \left(\frac{1}{n} - \frac{1}{N}\right) S_y^2 \), \( \text{Var}(\bar{x}) = \left(\frac{1}{n} - \frac{1}{N}\right) S_x^2 \), \( \text{Cov}(\bar{x}, \bar{y}) = \left(\frac{1}{n} - \frac{1}{N}\right) S_{yx} \), and \( R = \frac{\bar{Y}}{\bar{X}} \), and \( S_x^2 \), \( S_y^2 \) denote the population variance of auxiliary variable and study variable respectively and \( S_{yx} \) is the population covariance between auxiliary variable and study variable [ see Cochran(1977)].
\[
\text{Bias}(\overline{y}_{ss}) = \left( \frac{1}{n} - \frac{1}{N} \right) \overline{Y} \left[ \frac{1 + \alpha \frac{S_y^2}{X^2}}{2} - \alpha \frac{S_{yx}}{XY} \right]
\]  
(4)

and
\[
\text{MSE}(\overline{y}_{ss}) = \left( \frac{1}{n} - \frac{1}{N} \right) \left[ S_y^2 + \alpha^2 R^2 S_x^2 - 2\alpha RS_{yx} \right]
\]  
(5)

For large value of \( N \), we have
\[
\text{Bias}(\overline{y}_{ss}) = \frac{\overline{Y}}{n} \left[ \frac{1 + \alpha \frac{S_y^2}{X^2}}{2} - \alpha \frac{S_{yx}}{XY} \right]
\]  
(6)

and
\[
\text{MSE}(\overline{y}_{ss}) = \frac{1}{n} \left[ S_y^2 + \alpha^2 R^2 S_x^2 - 2\alpha RS_{yx} \right]
\]  
(7)

The optimum value of \( \alpha \), which minimizes the \( \text{MSE}(\overline{y}_{ss}) \) can be obtained after solving the following equation
\[
\frac{\partial \text{MSE}(\overline{y}_{ss})}{\partial \alpha} = 0.
\]

The optimum value of \( \alpha \) is
\[
\alpha^{opt} = \text{Cov}(\overline{x}, \overline{y}) = \frac{S_{yx}}{RV(x)} = \frac{\beta}{R} = \alpha'(\text{say}),
\]
where \( \beta = \frac{S_{yx}}{S_x^2} \), regression coefficient \( y \) on \( x \).

After using \( \alpha' \) in (5), we get the minimum \( \text{MSE}(\overline{y}_{ss}) \) as
\[
\text{MSE}_{\text{min}}(\overline{y}_{ss}) = \left( \frac{1}{n} - \frac{1}{N} \right) (1 - \rho^2) S_y^2.
\]  
(8)

Sahai and Sahai (1985) proved that at \( \alpha' \), \( \overline{y}_{ss} \) is always more efficient than classical ratio estimator and mean per unit estimator of population mean.

Since the value of \( \alpha' \) depends upon population parameters, so a reliable value of \( \alpha' \) is not available in some practical situations. In such cases we estimate \( \alpha' \) from sample data as
\[
\hat{\alpha}' = \frac{S_{yx}}{RS_x^2}
\]
where $s_{x}^{2}$, $s_{yx}$ are sample variance of auxiliary variable and sample covariance between auxiliary and study variables respectively and $\hat{R} = \frac{\bar{y}}{\bar{x}}$.

The modified estimator may be defined as

$$
\hat{y}_{SS} = \bar{y} = \frac{(1+\hat{\alpha})\bar{X} + (1-\hat{\alpha})\bar{x}}{(1-\hat{\alpha})\bar{X} + (1+\hat{\alpha})\bar{x}}.
$$

1.3 Under RSS

Let $(y_{j(i)}, x_{j(i)})$ denote an $i^{th}$ unit in the $j^{th}$ ranked set sample, where $i = 1, 2, ..., n'$; $j = 1, 2, ..., m$ and $mn' = n$.

Jemain et al. (2008) defined a ratio estimator of population mean using RSS as

$$
\hat{y}_{JRSS} = \bar{y}_{RSS} \left( \frac{\bar{X} + \rho}{\bar{x}_{RSS} + \rho} \right),
$$

(9)

where $\bar{y}_{RSS} = \frac{1}{mn'} \sum_{j=1}^{m} \sum_{i=1}^{n'} y_{j(i)}$ and $\bar{x}_{RSS} = \frac{1}{mn'} \sum_{j=1}^{m} \sum_{i=1}^{n'} x_{j(i)}$.

The bias and MSE of this estimator can be given by

$$
\text{Bias}(\hat{y}_{JRSS}) = \bar{y} \left[ \frac{V(\bar{x}_{RSS})}{(\bar{X} + \rho)^2} - \frac{\text{Cov}(\bar{x}_{RSS}, \bar{y}_{RSS})}{\bar{Y}(\bar{X} + \rho)} \right],
$$

(10)

and

$$
\text{MSE}(\hat{y}_{JRSS}) = \bar{y} \left[ \frac{V(\bar{y}_{RSS})}{\bar{Y}^2} + \frac{V(\bar{x}_{RSS})}{(\bar{X} + \rho)^2} - \frac{2\text{Cov}(\bar{x}_{RSS}, \bar{y}_{RSS})}{\bar{Y}(\bar{X} + \rho)} \right].
$$

(11)

where

$$
V(\bar{y}_{RSS}) = \frac{1}{mn'} \left( S_{y}^2 - \frac{1}{n'} \sum_{i=1}^{n'} S_{y(i)}^2 \right),
$$

(12)

$$
V(\bar{x}_{RSS}) = \frac{1}{mn'} \left( S_{x}^2 - \frac{1}{n'} \sum_{i=1}^{n'} S_{x(i)}^2 \right),
$$

(13)

and

$$
\text{Cov}(\bar{x}_{RSS}, \bar{y}_{RSS}) = \frac{1}{mn'} \left( S_{yx} - \frac{1}{n'} \sum_{i=1}^{n'} S_{y(i) x(i)} \right),
$$

(14)
where \( \tau_{x(i)} = \mu_{x(i)} - \bar{X} \), \( \tau_{y(i)} = \mu_{y(i)} - \bar{Y} \) and \( \tau_{y(i)}(i) = (\mu_{y(i)} - \bar{Y})(\mu_{x(i)} - \bar{X}) \). The values of \( \mu_{x(i)} \) and \( \mu_{y(i)} \) can be found in Arnold et al. (1992) which depend on order statistics from some specific distributions. Here the values of \( \mu_{x(i)} \) and \( \mu_{y(i)} \) can be taken to be same in the absence of judgment errors if the variables have the same distribution [see Dell and Clutter (1972)].

Using (12), (13), and (14) in (11), we get

\[
\text{MSE}(\bar{y}_{\text{RSS}}) = \frac{1}{mn} \left[ S_y^2 + \frac{\bar{Y}^2}{(X + \rho)^2} S_x^2 - 2 \frac{\bar{Y}}{(X + \rho)} S_{yx} \right] - \frac{1}{mn^2} \sum_{i=1}^{n} \left( \tau_{y(i)} - \frac{\bar{Y}}{(X + \rho)} \tau_{x(i)} \right)^2. \quad (15)
\]

Jemain et al. (2008) have shown that the estimator \( \bar{y}_{\text{RSS}} \) is more efficient than the classical ratio estimator under RSS [Samawi and Muttlak (1996)].

Bouza (2008) proposed a product estimator of population mean under RSS given as

\[
\bar{y}_{\text{PrSS}} = \bar{y}_{\text{RSS}} \frac{x_{\text{RSS}}}{X}. \quad (16)
\]

Bias and approximate MSE of \( \bar{y}_{\text{PrSS}} \) are as follows:

\[
\text{Bias}(\bar{y}_{\text{PrSS}}) = \frac{\text{Cov}(x_{\text{RSS}}, \bar{y}_{\text{RSS}})}{X}. \quad (17)
\]

and

\[
\text{MSE}(\bar{y}_{\text{PrSS}}) = V(\bar{y}_{\text{RSS}}) + R^2V(\bar{x}_{\text{RSS}}) + 2RV(x_{\text{RSS}}, \bar{y}_{\text{RSS}}). \quad (18)
\]

where the values of \( V(\bar{y}_{\text{RSS}}) \), \( V(\bar{x}_{\text{RSS}}) \) and \( \text{Cov}(x_{\text{RSS}}, \bar{y}_{\text{RSS}}) \) are given in (12), (13), and (14) respectively. \( \bar{y}_{\text{PrSS}} \) is more efficient than \( \bar{y}_{\text{RSS}} \) when correlation coefficient \((\rho)\) between study variable and auxiliary variable is negative.

2. PROPOSED ESTIMATOR

We propose an estimator which can use auxiliary information more efficiently as given below

\[
\bar{y}_{\text{BMRSS}} = \bar{y}_{\text{RSS}} \frac{(1 + \alpha)X + (1 - \alpha)x_{\text{RSS}}}{(1 - \alpha)X + (1 + \alpha)x_{\text{RSS}}} \quad (19)
\]

where \( \alpha \) is a pre-assigned constant.
The approximate expression for Bias and MSE of $\bar{y}_{BMrss}$ are as

$$
Bias(\bar{y}_{BMrss}) = \bar{Y} \left[ \frac{1 + \alpha}{2} \frac{\alpha}{X^2} \right] - \bar{Y} \frac{\alpha}{XY} \left[ Cov(\bar{x}_{rss}, \bar{y}_{rss}) \right]
$$

(20)

and

$$
MSE(\bar{y}_{BMrss}) = V(\bar{y}_{rss}) + \alpha^2 \sigma^2 + 2\sigma R Cov(\bar{x}_{rss}, \bar{y}_{rss}).
$$

(21)

After using equations (12), (13) and (14) in (20) and (21), we get

$$
Bias(\bar{y}_{BMrss}) = \bar{Y} \left[ \frac{1 + \alpha}{2} \frac{S_{xy}^2}{X^2} - \frac{S_{yx}}{XY} \right] - \bar{Y} \frac{\alpha}{XY} \frac{R}{n} \sum_{i=1}^{n} (\bar{x}_{x(i)} - \alpha) \frac{\tau_{x(i)}^2}{X^2}
$$

(22)

and

$$
MSE(\bar{y}_{BMrss}) = \frac{1}{mn} \left[ S_{xy}^2 + \alpha^2 \sigma^2 + 2\sigma R S_{xy} \right] - \frac{1}{mn} \frac{R}{n} \sum_{i=1}^{n} (\bar{x}_{y(i)} - \alpha) \tau_{x(i)}^2.
$$

(23)

One can obtain the optimum value of $\alpha$, which minimizes the $MSE(\bar{y}_{BMrss})$ by solving the following equation

$$
\frac{\partial MSE(\bar{y}_{BMrss})}{\partial \alpha} = 0.
$$

The optimum value of $\alpha$ is

$$
\alpha^{opt} = \frac{Cov(\bar{x}_{rss}, \bar{y}_{rss})}{RV(\bar{x}_{rss})} = \frac{S_{xy} - \frac{1}{n} \sum_{i=1}^{n} \tau_{y(i)}}{RV(\bar{x}_{rss})} = \alpha^* \text{(say)}
$$

After using the optimum value of $\alpha$ in (22), we get

$$
MSE_{min}(\bar{y}_{BMrss}) = \frac{1}{1 - \rho_{x_{rss}, y_{rss}}} V(\bar{y}_{rss})
$$

(24)

where $\rho_{x_{rss}, y_{rss}}$ is the correlation coefficient between $\bar{x}_{rss}$ and $\bar{y}_{rss}$.

Here it is to be noted that the value of $\alpha^*$ depends upon population parameters, so value of $\alpha^*$ is generally not available in practical problems. We estimate this value from sample data as follows:
\begin{equation}
\hat{\alpha}^* = \frac{\sum_{i=1}^{m} s_{y(i)} x_{y(i)} - \bar{X}}{\sum_{i=1}^{m} s_{x(i)}^2 - y},
\end{equation}

where \( s_{y(i)} = \frac{1}{m-1} \sum_{j=1}^{m} (x_{j(i)} - \bar{x}_{(i)}) (y_{j(i)} - \bar{y}_{(i)}) \) and \( s_{x(i)}^2 = \frac{1}{m-1} \sum_{j=1}^{m} (x_{j(i)} - \bar{x}_{(i)})^2 \). Here \( \bar{x}_{(i)} \) and \( \bar{y}_{(i)} \) denote the sample means for \( i^{th} \) order statistics of auxiliary and study variables respectively.

The modified estimator may be defined as

\begin{equation}
\tilde{y}_{Bmrs} = \frac{\alpha^{*} (1 + \alpha^{*}) \bar{X} + (1 - \alpha^{*}) \bar{y}_{rss}}{(1 - \alpha^{*}) \bar{X} + (1 + \alpha^{*}) \bar{y}_{rss}}.
\end{equation}

3. COMPARISON

\( \tilde{y}_{Bmrs} \) v/s \( \tilde{y}_{rss} \)

From (11) and (24), we have

\begin{equation}
MSE(\tilde{y}_{rss}) - MSE_{\min}(\tilde{y}_{Bmrs}) = \left( \rho_{y_{rss}, \gamma_{rss}} \sqrt{V(\tilde{y}_{rss})} - \frac{\bar{Y}}{\bar{X} + \rho} \sqrt{V(\bar{y}_{rss})} \right)^2 \geq 0,
\end{equation}

which shows that \( \tilde{y}_{Bmrs} \) is always more efficient than \( \tilde{y}_{rss} \).

\( \tilde{y}_{Bmrs} \) v/s \( \tilde{y}_{Prs} \)

From (18) and (24), we have

\begin{equation}
MSE(\tilde{y}_{Prs}) - MSE_{\min}(\tilde{y}_{Bmrs}) = \left( \rho_{y_{Prs}, \gamma_{Prs}} \sqrt{V(\tilde{y}_{Prs})} - \sqrt{V(\bar{y}_{rss})} \right)^2 \geq 0,
\end{equation}

which shows that \( \tilde{y}_{Bmrs} \) is always more efficient than \( \tilde{y}_{Prs} \).

\( \tilde{y}_{Bmrs} \) v/s \( \tilde{y}_{SS} \)

At any value of \( \alpha \), from (7) and (23), we have

\begin{equation}
MSE_{\alpha}(\tilde{y}_{SS}) - MSE_{\alpha}(\tilde{y}_{Bmrs}) = \frac{1}{mn} \sum_{i=1}^{m} (\tau_{y(i)} - \alpha \tau_{x(i)})^2 \geq 0.
\end{equation}
This implies, at $\alpha'$, we get

$$MSE_{\alpha'}(\bar{y}_{SS}) \geq MSE_{\alpha'}(\bar{y}_{BMres}).$$

Since $\alpha'$ is the optimum value of $\alpha$ which minimize the MSE of $\bar{y}_{BMres}$, that imply

$$MSE_{\alpha'}(\bar{y}_{BMres}) \geq MSE_{\alpha'}(\bar{y}_{BMres}).$$

From above two inequalities, we get

$$MSE_{\alpha'}(\bar{y}_{SS}) \geq MSE_{\alpha'}(\bar{y}_{BMres}).$$

This shows that $\bar{y}_{BMres}$ is always more efficient than $\bar{y}_{SS}$.

4. SIMULATION STUDY

To verify theoretical results of the proposed estimator, we perform a simulation study by using software R. 1,00,000 samples are drawn from bivariate normal distribution BVN (200, 100, 4, 4, $\rho$), for $\rho = -0.9, -0.7, -0.5, -0.3, 0.3, 0.5, 0.7, 0.9$, under RSS design and SRS design.

The efficiency of an estimator $\hat{\theta}$ with respect to $\bar{y}$ to estimate population mean $\bar{Y}$ is defined as:

$$eff(\hat{\theta}) = \frac{MSE(\bar{y})}{MSE(\hat{\theta})}.$$ 

The bias and MSE values of $\bar{y}, \bar{y}_p, \bar{y}_r, \bar{y}_{SS}, \bar{y}_{SS}, \bar{y}_{Prss}, \bar{y}_{Rres}, \bar{y}_{Jres}, \bar{y}_{BMres}$ and $\bar{y}^{p}_{BMres}$ are obtained for $n = 9, 12, 15$, where $n' = 3, 4, 5$ and $m = 3$ in Table-1 and Table-2 respectively.

The efficiencies of $\bar{y}, \bar{y}_p, \bar{y}_r, \bar{y}_{SS}, \bar{y}_{SS}, \bar{y}_{Prss}, \bar{y}_{Rres}, \bar{y}_{Jres}, \bar{y}_{BMres}$ and $\bar{y}^{p}_{BMres}$ with respect to $\bar{y}$ for different values of $\rho$ and $n$, are shown in Table-3.

4.1 Some Results Based On Simulation Study

Some conclusions are drawn from the Table-3 as follow:

1. For $\rho > 0$, 

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\[
\text{eff}(y_p) < \text{eff}(y_{PSS}) < \text{eff}(y_R) < \text{eff}(y_{RSS}) < \text{eff}(y_{SS}) < \text{eff}(y_{BMSS}).
\]

2. For \( \rho < 0 \),
\[
\text{eff}(y_R) < \text{eff}(y_{RSS}) < \text{eff}(y_{PSS}) < \text{eff}(y_p) < \text{eff}(y_{PSS}) < \text{eff}(y_{BMSS}).
\]

3. As \(|\rho|\) increases, efficiency of the proposed estimator increases.

4. As the sample size \( n \) increases, efficiency of the proposed estimator also increases.

5. In the situation when we estimate the optimum value of \( \alpha \) from sample data, there is some loss of efficiency but \( \bar{y}_{BMSS} \) is still more efficient than the other estimators.

5. CONCLUSION

The simulation study reveals that the proposed ratio-type estimator of population mean is more efficient than the estimator suggested by Sahai and Sahai (1985). Thus we can say that the proposed estimator of population mean is more efficient as compared to the estimator developed under SRS design, for any value of \( \alpha \). Also, the proposed estimator is more efficient than the estimators defined by Samawi and Muttlak (1996), Bouza (2008) and Jemain et al. (2008). Hence, the use of proposed estimator of population mean would be more beneficial than the existing ones.

Table 1.

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( B(y) )</th>
<th>( B(y_p) )</th>
<th>( B(y_R) )</th>
<th>( B(y_{SS}) )</th>
<th>( B(y_{PSS}) )</th>
<th>( B(y_{RSS}) )</th>
<th>( B(y_{BMSS}) )</th>
<th>( B(y_{BMSS}) )</th>
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<td>-</td>
<td>0.0127</td>
<td>-</td>
<td>-</td>
<td>-</td>
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<td>0.0059</td>
</tr>
<tr>
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<td>0.0101</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0.0002</td>
<td>0.0018</td>
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<tr>
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<td>0.0080</td>
<td>0.0003</td>
<td>0.0005</td>
<td>0.0031</td>
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<td>0.0064</td>
<td>0.0007</td>
<td>0.0007</td>
<td>0.0041</td>
<td>0.0046</td>
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<td>0.0077</td>
<td>0.0027</td>
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<td>0.9</td>
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<td>0.0017</td>
<td>0.0047</td>
<td>0.0010</td>
<td>0.0009</td>
</tr>
</tbody>
</table>

\( n = 9, m = 3, \) \( n = 12, n = 4, m = 3 \)
Table 2.

MSEs of \( y, \hat{y}_R, y_{SS}, y_{SSS}, y_{P_{SS}}, y_{PR_{SS}}, y_{PR_{SSS}}, y_{BM_{SSS}} \) and \( \hat{y}_P \):

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( MSE(y) )</th>
<th>( MSE(y_R) )</th>
<th>( MSE(y_{SS}) )</th>
<th>( MSE(y_{SSS}) )</th>
<th>( MSE(y_{PSS}) )</th>
<th>( MSE(y_{PRSS}) )</th>
<th>( MSE(y_{PRSSS}) )</th>
<th>( MSE(y_{BMSSS}) )</th>
<th>( MSE(y_P) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.9</td>
<td>0.444</td>
<td>0.622</td>
<td>3.824</td>
<td>0.084</td>
<td>0.098</td>
<td>0.481</td>
<td>2.135</td>
<td>2.162</td>
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</tr>
<tr>
<td>-0.7</td>
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<td>0.978</td>
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<td>0.227</td>
<td>0.264</td>
<td>0.934</td>
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<td>2.244</td>
<td>0.152</td>
</tr>
<tr>
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<td>2.201</td>
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</tr>
<tr>
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<td>1.690</td>
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<td>2.217</td>
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<td>0.623</td>
<td>0.084</td>
<td>0.098</td>
<td>2.132</td>
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<td>0.071</td>
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Table 3.

Efficiencies of \( y, \hat{y}_R, y_{SS}, y_{SSS}, y_{P_{SS}}, y_{PR_{SS}}, y_{PR_{SSS}}, y_{BM_{SSS}} \) and \( \hat{y}_P \):

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( \text{eff}(y) )</th>
<th>( \text{eff}(y_R) )</th>
<th>( \text{eff}(y_{SS}) )</th>
<th>( \text{eff}(y_{SSS}) )</th>
<th>( \text{eff}(y_{PSS}) )</th>
<th>( \text{eff}(y_{PRSS}) )</th>
<th>( \text{eff}(y_{PRSSS}) )</th>
<th>( \text{eff}(y_{BMSSS}) )</th>
<th>( \text{eff}(y_P) )</th>
</tr>
</thead>
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<td>0.714</td>
<td>0.116</td>
<td>5.254</td>
<td>4.521</td>
<td>0.923</td>
<td>0.208</td>
<td>0.205</td>
<td>6.2520</td>
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<tr>
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<td>0.128</td>
<td>1.958</td>
<td>1.678</td>
<td>0.476</td>
<td>0.200</td>
<td>0.198</td>
<td>2.9109</td>
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<tr>
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<td>0.142</td>
<td>1.331</td>
<td>1.138</td>
<td>0.337</td>
<td>0.198</td>
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<tr>
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<td>0.262</td>
<td>0.161</td>
<td>1.098</td>
<td>0.936</td>
<td>0.271</td>
<td>0.203</td>
<td>0.201</td>
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<td>0.161</td>
<td>0.262</td>
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<td>0.924</td>
<td>0.947</td>
<td>6.2443</td>
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</table>

| \( \rho \) | \( \text{eff}(y) \) | \( \text{eff}(y_R) \) | \( \text{eff}(y_{SS}) \) | \( \text{eff}(y_{SSS}) \) | \( \text{eff}(y_{PSS}) \) | \( \text{eff}(y_{PRSS}) \) | \( \text{eff}(y_{PRSSS}) \) | \( \text{eff}(y_{BMSSS}) \) |
|---------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| -0.9    | 1           | 0.714      | 0.116      | 5.281      | 4.754      | 0.973      | 0.247      | 0.244      | 6.6708      | 5.7360      |
| -0.7    | 1           | 0.455      | 0.128      | 1.966      | 1.764      | 0.475      | 0.224      | 0.222      | 3.3603      | 2.8812      |
| -0.5    | 1           | 0.334      | 0.142      | 1.335      | 1.198      | 0.333      | 0.214      | 0.212      | 2.7307      | 2.3357      |
| -0.3    | 1           | 0.264      | 0.161      | 1.099      | 0.986      | 0.270      | 0.212      | 0.211      | 2.4950      | 2.1327      |
| 0.3     | 1           | 0.161      | 0.263      | 1.098      | 0.991      | 0.212      | 0.270      | 0.271      | 2.4928      | 2.1299      |
| 0.5     | 1           | 0.143      | 0.334      | 1.333      | 1.203      | 0.214      | 0.333      | 0.337      | 2.7267      | 2.3327      |
| 0.7     | 1           | 0.128      | 0.456      | 1.961      | 1.765      | 0.225      | 0.475      | 0.482      | 3.3536      | 2.8752      |
| 0.9     | 1           | 0.116      | 0.716      | 5.272      | 4.723      | 0.248      | 0.975      | 0.998      | 6.6570      | 5.7326      |
REFERENCES