Estimation of Parameters and Reliability Characteristics in Self Relocated Design

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Abstract

In this paper self-relocated experimental design introduced by Srivastava [1987] has been studied when the life-times of the systems follow generalized exponential distribution. Newton-Raphson method is adopted to obtain the maximum likelihood estimators of the unknown parameters, and the associated standard error. Extensive simulations experiments are conducted to study the performances of the proposed estimators, and also to verify the small sample behaviour of the estimators based on asymptotic results. Further, we study the optimality of the self-relocated design under generalized type-II censoring scheme. Finally, some cost analysis based on simulation study has been carried out.

Keywords: Asymptotic Variance- Covariance, Generalized Exponential Distribution, ML Estimator, Monte-Carlo Simulation Technique, Newton-Raphson Method, Self Relocated Design

AMS Subject Classification: 62N05, 62K99

1. Introduction

The industrial revolution onwards man has become more and more dependent on “Machine” or “System”. The failure of the machine stops the whole system abruptly. Therefore, it is important to know among the several machines doing the same work, which machine should be faithful or reliable for their daily usage and having minimum maintenance but not unduly expensive. One of the objectives that may be adopted is to select the system having high reliability. This requires results from extensive reliability and life testing experiments. Most commonly used sampling designs in these experiments are type I censoring design (TICD), type II censoring design (TIICD) and random censoring design (RCD) (Balakrishnan and Aggrawala [2012]). To address the problems of reliability of complex systems, Srivastava [1986] had introduced the concept of “Reliability-Oriented-Design” theory, a branch of the general field of the statistical design of the scientific experiments. In this pursuit first he had introduced a new class of designs called Self Relocated Designs (SRD) for comparative experiments in the reliability field, and studied a
subclass of these in detail under exponential distribution. Srivastava [1987, 1989] had studied the same scheme by assuming lifetimes to follow generalized Weibull distribution. Further, he had shown that in general, the scheme improves accuracy and reduces the total expected time under experiment as well as the expected length of the experiment. One may continue the study of SRD by considering gamma distribution as it also includes IFR and DFR life distributions as the members. However, use of gamma distribution is not encouraging as its cumulative distribution is not in the closed form. Recently, Gupta and Kundu [1999] have proposed generalized exponential (GE) distribution which has some common properties of both gamma and Weibull distributions. Further, it enhances its utility for having closed form expression for its cumulative distribution function. There are many properties of the distribution discussed in Gupta and Kundu [1999]. Several authors have worked on GE distribution under various censoring schemes and studied their properties as well as their applications. Nadarajah, S. (2011) provides complete review of GE distribution and related study since its inception. Looking at versatility of the distribution and wide scale applicability, we study the SRD under generalized exponential life distribution. We denote our experimental scheme by SRD21GED. The whole organization of the paper is as below:

In Section 2 we briefly introduce Self Relocated Design of Srivastava [1987]. In Section 3 we give the probability density function, the survival function and the hazard rate of the generalized exponential distribution. We also develop likelihood of SRD21GED. In Section 4 we derive the expressions for maximum likelihood estimators of parameters and their asymptotic variance-covariance matrix when shape parameter of the distribution is known and when it is unknown. Section 5 discusses algorithm for generation of data from SRD21GED and provides iterative procedure for estimation of the parameters through Newton-Raphson method. Further, the tables of ML estimates and their asymptotic standard errors, estimate of reliability and hazard rates and their mean square error at fixed time point are given which are simulated through Monte-Carlo simulation technique for both the cases of shape parameter known and unknown. In Section 6, we discuss likelihood ratio test for simultaneous testing of homogeneity of scale parameters when the shape parameter is known. The cut-off points for the test statistics are obtained through Monte-Carlo simulation. In Section 7, we simulate design optimality criteria for SRD and generalized type II censoring design. In Section 8, cost functions for the problem of planning such experiments are discussed. Some concluding remarks are given in Section 9.
2. Self Relocated Design (SRD)

Consider \( m \) systems of same “capacity” made for the same purpose but of different brands or make. Here we are interested in studying the reliability of these systems for the better choice. This requires some designed experiment which will be terminated at the earlier time such that the life time data provides some meaningful information for the better choice of the system. Many experimental schemes are available in statistics literature like Type I censoring or time censoring, Type II censoring or item censoring, random censoring, progressive censoring, hybrid censoring etc. But most of these censoring schemes are proposed for the study of single population or comparison of two populations. There is very sparse literature on generalization of these concepts to multiple populations. Srivastava [1986], see also Srivastava [1987, 1989], first introduced the SRD mainly for comparison of multiple populations. For completeness purposes, we briefly describe SRD below.

Label the \( m \) brands as \( 1, 2, \ldots, m \). In the identical environment, suppose \( u \) items of each brand are put on a life testing experiment. As and when an unit fails the brand of the corresponding unit is noted, and failure time is also recorded. Further, at the time failure one unit each from the other brands, those have not failed, is randomly dropped so that number units of each brand in the life testing experiment remain constant at any point of time. The experiment is continued till \( G \) failures, \( G \) is utmost equal \( u \), are observed. Schematically the design is as shown below.

Let us consider \( (t_g, i_g) \) where \( t_g \) denote the \( g \)-th failure, \( i_g \) denote the label of the brand of system failed, \( g=1,2,\ldots,G \).

The above design is “self-relocating” in the sense that the items being tested at a particular time depend upon the development of the (life) test itself. This design is suggested by Srivastava [1986] and denoted it by SRD21, the 2 in 21 denoting the fact the number of items being tested is decreasing with time, and 1 in 21 indicating removal of one unit each
from those brands from which unit did not fail at that time. In this paper authors consider SRD21GED model, which means we are considering the design SRD21 with the further assumption that the life time obey the "Generalized Exponential Distribution" (denoted by GED), as discussed in the next section.

3. Development of Likelihood Function for SRD21GED under Generalized Exponential Distribution

3.1 Generalized Exponential Distribution

Consider an item whose life time is denoted by \( T \). The random variable \( T \) is assumed to have generalized exponential distribution (GE), as defined by Gupta and Kundu [1999], with distribution function

\[
F(t; \alpha, \beta) = \left(1 - e^{-\beta t}\right)^\alpha \quad (t > 0, \alpha > 0, \beta > 0).
\]  

(3.1)

The corresponding density function is given by

\[
f(t; \alpha, \beta) = \alpha \beta (1 - e^{-\beta t})^{\alpha-1} e^{-\beta t} \quad (t > 0, \alpha > 0, \beta > 0).
\]

(3.2)

Here \( \alpha \) is shape parameter, \( \beta \) is a scale parameter. We denote the GE distribution with shape parameter \( \alpha \) and scale parameter \( \beta \) as GE (\( \alpha, \beta \)).

Then the reliability function is

\[
\bar{F}(t) = P(T > t) = 1 - \left(1 - e^{-\beta t}\right)^\alpha
\]

(3.3)

and the hazard function is

\[
h(t) = \alpha \beta \left[ \frac{e^{-\beta t}}{1 - e^{-\beta t}} \right] \left[ \frac{(1 - e^{-\beta t})^\alpha}{1 - (1 - e^{-\beta t})^\alpha} \right].
\]

(3.4)

If \( Y \) follows GE(\( \alpha, 1 \)), then the corresponding moment generating function, is given by

\[
M(s) = \alpha \int_0^\infty (1 - e^{-y})^{\alpha-1} e^{(y-1)s} dy
\]

\[
= \alpha \int_0^1 (1 - z)^{\alpha-1} z^{-1} dz = \frac{\Gamma(\alpha+1) \Gamma(1-s)}{\Gamma(\alpha-s+1)} \quad s < 1
\]

(3.5)

Differentiating ln\( M(s) \) and evaluating at \( s = 0 \), we get the mean and variance of GE(\( \alpha, 1 \)) as

\[
E(Z) = \psi(\alpha + 1) - \psi(1) \quad \text{and} \quad \text{var}(Z) = \psi'(1) - \psi'(\alpha + 1)
\]

(3.6)

where \( \psi(\cdot) \) is the digamma function and \( \psi'(\cdot) \) is its derivative.
If \( Z \) follows \( GE(\alpha, 1) \), and \( T = \frac{1}{\beta} Z \) then \( T \) follows \( GE(\alpha, \beta) \). Therefore, the mean and variance of \( T \) is given by

\[
E(T) = \frac{\psi(\alpha+1) - \psi(1)}{\beta} \quad \text{and} \quad \text{var}(T) = \frac{\psi(1) - \psi'(\alpha+1)}{\beta^2}.
\] (3.7)

### 3.2 Likelihood Function for SRD21GED

The contribution of the observed system lifetimes in the likelihood can be studied as follows. Let us consider the whole experiment divided into \( G \) stages, the time period for the \( g \)-th (\( g = 1, 2, \ldots, G \)) stage being \( (t_{g-1}, t_g] \). At time \( t_g \), \( g \)-th failure is observed from the systems of type \( i_g \), where \( i_g \) is an integer from 1 to \( m \). This leaves \( u - g + 1 \) surviving systems of type \( i_g \) at time \( t_g \) and remaining \( u - g \) surviving systems of each type \( i, i = 1, 2, \ldots, m; i \neq i_g \). Further, let \( l_g \) be the conditional likelihood for the \( g \)-th stage, given \( t_{g-1} \). Since at \( (t_{g-1} + 0) \), exactly \( u - g + 1 \) systems of each type is alive, we have

\[
l_g = \left( \frac{u - g + 1}{1} \right) \prod_{i=1}^{m} \left[ \frac{1}{1 - e^{-\beta_{ig} t_g}} \right]^{u-g+1} \prod_{i=1}^{m} \left[ \frac{1}{1 - e^{-\beta_{ig} t_g}} \right]^{u-g+1}
\] (3.8)

We shall assume shape parameters are same and scale parameters are different for the lifetimes of different brands which follow the distribution given in (3.2). Then using (3.2) in (3.3) and (3.8) we have,

\[
l_g = \left( \frac{u - g + 1}{1} \right) \alpha \beta_{ig} \left[\frac{e^{-\beta_{ig} t_g}}{1 - e^{-\beta_{ig} t_g}}\right]^{u-g+1} \prod_{i=1}^{m} \left[ \frac{1}{1 - e^{-\beta_{ig} t_g}} \right]^{u-g+1}
\] (3.9)

Let \( L_g \) be the likelihood for the whole experiment, we must have

\[
L_G = \prod_{g=1}^{G} l_g
\]

Therefore, the simplified form of \( L_G \) can be derived as:

\[
L_G = l_1 \cdot l_2 \cdot \ldots \cdot l_G
\]

\[
= \frac{(u)}{1} \frac{(u-1)}{1} \ldots \frac{(u-g+1)}{1} \ldots \frac{(u-G+1)}{1} \alpha \prod_{g=1}^{G} \beta_{ig}
\]

\[
\left( \frac{e^{-\beta_{i_1} t_1}}{1 - e^{-\beta_{i_1} t_1}} \right) \left( \frac{(1 - e^{-\beta_{i_1} t_1})^\alpha}{1 - (1 - e^{-\beta_{i_1} t_1})^\alpha} \right) \left( \frac{e^{-\beta_{i_2} t_2}}{1 - e^{-\beta_{i_2} t_2}} \right) \left( \frac{(1 - e^{-\beta_{i_2} t_2})^\alpha}{1 - (1 - e^{-\beta_{i_2} t_2})^\alpha} \right) \ldots
\]
Define for and .

Substitute where and in equation (3.10), we have

4. Maximum Likelihood Estimation

In this section we obtain maximum likelihood estimates of \( \alpha, \beta_i \) \((i = 1, 2, \ldots, m)\), reliability function, hazard rate and observed Information Matrix under the design.

Using the likelihood function (3.12) the log likelihood function can be written as

\[
\ln L_G = \ln \left( \frac{u!}{(u-G)!} \right) + Gl u + \sum_{i=1}^{m} \delta_i \ln \beta_i - \sum_{g=1}^{G} \sum_{i=1}^{m} (\beta_i \delta_{ig}) t_g \\
- \sum_{g=1}^{G} \ln \left[ 1 - e^{-\left( \sum_{i=1}^{m} \beta_i \delta_{ig} \right) t_g} \right] + \alpha \sum_{g=1}^{G} \ln \left[ 1 - e^{-\left( \sum_{i=1}^{m} \left( \beta_i \delta_{ig} \right) t_g \right)} \right] \\
- \sum_{g=1}^{G} \ln \left[ 1 - \left( 1 - e^{-\left( \sum_{i=1}^{m} \beta_i \delta_{ig} \right) t_g} \right) \right] + \sum_{i=1}^{m} \sum_{g=1}^{G} \ln \left[ 1 - \left( 1 - e^{-\left( \beta_i t_g \right)} \right) \right] + (u-G) \sum_{i=1}^{m} \sum_{g=1}^{G} \ln \left[ 1 - \left( 1 - e^{-\left( \beta_i t_g \right)} \right) \right] \tag{4.1}
\]

Differentiate (4.1) with respect to \( \alpha \) and \( \beta_i \) \((i = 1, 2, \ldots, m)\) we have
The estimates of parameters are obtained in two cases: (i) shape parameter is known and (ii) shape parameter is unknown.

4.1 Maximum Likelihood Estimation when $\alpha$ is Known

The solutions of equation (4.3) can be evaluated numerically by some suitable iterative procedure such as Newton-Raphson method, for given values of $\alpha$. The MLE of $\beta_i$s are obtained as from equations (4.3). The MLEs of Reliability ($R_i(t_i); i=1,2,\ldots,m$) and hazard rate ($h_i(t_i); i=1,2,\ldots,m$) can be evaluated using invariance property of MLEs as

\begin{align*}
\hat{R}_i(t_i) &= 1 - (1 - e^{-\hat{\beta}_i t_i})^\alpha \\
\hat{h}_i(t_i) &= \alpha \hat{\beta}_i \left[ \frac{e^{\hat{\beta}_i t_i}}{1 - e^{\hat{\beta}_i t_i}} \right]^\alpha \left[ \frac{(1 - e^{-\hat{\beta}_i t_i})^\alpha}{1 - (1 - e^{-\hat{\beta}_i t_i})^\alpha} \right] \quad \text{for } i = 1,2,\ldots,m
\end{align*}

4.1.1 Observed Fisher Information Matrix under the Design

To obtain Fisher Information Matrix we take derivatives of equation (4.3) with respect to $\beta_i$; $i = 1,2,\ldots,m$. Therefore we have,
Rate of failures of systems are independent of each type of systems, derivates of equation (4.3) with respect to $\beta_j; j \neq i = 1,2,\ldots, m$ are

\[
\frac{\partial^2 \ln L_G}{\partial \beta_i \partial \beta_j} = 0 \quad \forall j \neq i = 1,2,\ldots, m
\]  

For $\alpha > 2$, the Generalized Exponential family satisfies all the regularity conditions (See Bain, 1978, pp.86-87) in a similar way to the gamma family and the Weibull family, and therefore, we have the following results.

**Theorem 4.1:** For $\alpha > 2$ and $\frac{G}{u}$ kept constant the maximum likelihood estimators $\hat{\beta}$ of $\beta$ are consistent estimators, and $\sqrt{u}(\hat{\beta} - \beta)$ is asymptotically $m$-variate normal with mean $0$ and variance covariance matrix $V^{-1}$, where $V$ is expected value of negative of second derivative matrix of log likelihood with respect to $\beta$.

Note: Since evaluation of expected value is cumbersome we will use sample information matrix $\hat{\Sigma}$ which, under usual regularity conditions, converges asymptotically to Fisher Information Matrix.

### 4.2. Maximum Likelihood Estimation when Shape Parameter $\alpha$ is Unknown

The solutions of equations (4.2-4.3) can be evaluated numerically by some suitable iterative procedure such as Newton-Raphson method, for given values of $(u, G, t_g; g = 1,2,\ldots, G)$. The MLE of $(\alpha, \beta)$ are obtained as $(\hat{\alpha}, \hat{\beta})$ from equations (4.2-4.3). The MLEs of Reliability $(R_i(t_i); i = 1,2,\ldots, m)$ and hazard rate $(h_i(t_i); i = 1,2,\ldots, m)$ can be evaluated using invariance property of MLEs as

\[
\hat{R}_i(t_i) = 1 - (1 - e^{-\hat{\beta}_i t_i})^\hat{\alpha}
\]  

for $i = 1,2,\ldots, m$  \hspace{1cm} (4.6)
\[ \hat{h}_i(t_i) = \frac{e^{-\beta_i t_i}}{1 - e^{-\beta_i t_i}} \left( \frac{1 - e^{-\beta_i t_i}}{1 - e^{-\beta_i t_i}} \right)^{2} \quad \text{for } i = 1, \ldots, m \]  

(4.9)

4.2.1. Observed Fisher Information Matrix under Design

To obtain Fisher Information Matrix we take derivatives of equation (4.2) and (4.3) with respect to \( \alpha, \beta_i; i = 1,2, \ldots, m \). Therefore we have,

\[ \frac{\partial^2 \ln L_G}{\partial \alpha^2} = -\frac{G}{\alpha^2} + \sum_{g=1}^{G} \ln \left\{ \frac{1 - e^{-\left(\sum_{j=1}^{m} \beta_j \delta_{ij} \right) t_g}}{1 - e^{-\beta_i t_g}} \right\} \]

\[ -\sum_{i=1}^{m} \sum_{g=1}^{G} \ln \left\{ \frac{(1 - e^{-\beta_i t_g})^2}{1 - e^{-\beta_i t_g}} \right\} \]

\[ - (u - G) \sum_{i=1}^{m} \ln \left\{ \frac{(1 - e^{-\beta_i t_g})^2}{1 - e^{-\beta_i t_g}} \right\} \quad \text{(4.10)} \]

\[ \frac{\partial^2 \ln L_G}{\partial \alpha \partial \beta_i} = \sum_{g=1}^{G} \left\{ \frac{\beta_{ig} t_g e^{-\left(\sum_{j=1}^{m} \beta_j \delta_{ij} \right) t_g}}{1 - e^{-\left(\sum_{j=1}^{m} \beta_j \delta_{ij} \right) t_g}} \right\} \]

\[ + \alpha \sum_{g=1}^{G} \left\{ \beta_{ig} t_g e^{-\left(\sum_{j=1}^{m} \beta_j \delta_{ij} \right) t_g} \left[ 1 - e^{-\left(\sum_{j=1}^{m} \beta_j \delta_{ij} \right) t_g} \right]^{a-1} \right\} \]

\[ \frac{1 - e^{-\left(\sum_{j=1}^{m} \beta_j \delta_{ij} \right) t_g}}{a} \quad \text{(4.13)} \]

Derivatives of equation (4.3) with respect to \( \beta_i; i = 1,2, \ldots, m \) and \( \beta_j; j \neq i = 1,2, \ldots, m \) are given in equations (4.6) and (4.7) respectively.

Therefore, we have the following results.

**Theorem 4.2:** For \( \alpha > 2 \) and \( \frac{G}{u} \) kept constant the maximum likelihood estimators \((\hat{\alpha}, \hat{\beta})\) of \((\alpha, \beta)\) are consistent estimators, and \( \sqrt{u}(\hat{\alpha} - \alpha, \hat{\beta} - \beta) \) is asymptotically \((m+1)\) multivariate normal with mean \((0,0)\) and variance covariance variance covariance matrix
$W^{-1}$, where $W$ is expected value of negative of second derivative matrix of log likelihood with respect to $(\alpha, \beta)$.

Note: As earlier case, we will use sample information matrix $\tilde{W}$ which, under usual regularity conditions, converges asymptotically to Fisher Information Matrix.

5. Algorithm, Numerical Exploration and Conclusions

In this Section, a Monte Carlo simulation study is conducted to compare the performance of the estimates developed in the previous sections. Maximum likelihood estimates are obtained for observations generated through Self Relocated Design when numbers of systems to be compared are 2 and 3 for known as well as unknown shape parameter having failure distribution $GED(\alpha, \beta_i); \ i = 1, 2, \ldots, m$. All calculations are performed on the $R$-language version- R.2.12.0.

5.1 Known Shape Parameter

In this section, we carry out simulation study for two sets of parameter values $m = 2, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3$ and for $m = 3, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4$. The simulation is carried out for different values of $u$ and $G$. We simulate 1000 samples for each case. The simulation results are summarized in Table I and Table II. The values $\alpha$ are taken larger than 2 as per the suggestion given in Gupta and Kundu [1999]. To carry out our objective we proceed through following algorithm.

Step 1: Taking $m = 2, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3$ we generate $u$ random numbers from $GE(\alpha, \beta_1, \beta_2, \ldots, \beta_m)$ for each type of systems. The same is repeated for the parameters $m = 3, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4$.

Step 2: Generate SRD observations using the technique discussed in Section 2. The generate observations are as below.

- The $G$ failure times $(t_1, t_2, \ldots, t_G)$.
- The system types those fail or those are withdrawn per the scheme at $t_1, t_2, \ldots, t_G$ are recorded in matrix of order $G \times m$: The $(i, j)$-th element is 1 if at time $t_i$ a $j$-th type system fails and is 0 if $j$-th system is withdrawn.

Step 3: Obtain initial estimate of parameters $\beta_i; i = 1, 2, \ldots, m$ by substituting $\alpha = 1$ in equation (4.3). For $\alpha = 1$ the maximum likelihood estimates of $\beta_i$'s are

$$\hat{\beta}_{i0} = \frac{\delta_i}{\sum_{g=1}^{G} \delta_g \tau_g + (u-G) \tau_G}, \ i = 1, 2, \ldots, m$$
Step 4: Obtain initial value of sample information matrix $\tilde{V}$ using the value obtained in Step 3 and also obtains the score vector $S' = \left( \frac{\partial l}{\partial \beta_1}, \frac{\partial l}{\partial \beta_2}, \ldots, \frac{\partial l}{\partial \beta_m} \right)$

Step 5: Use Newton-Raphson iterative method
$$\hat{\beta}_{New} = \hat{\beta}_{Old} + \tilde{V}^{-1} \left( \hat{\beta}_{Old} \right) * S$$

Step 6: Repeat the Step 5 until the $\sum_{i=1}^{m} |\hat{\beta}_{New} - \hat{\beta}_{Old}| < \epsilon$ where $\epsilon$ is very small predefined quantity.

Step 7: Repeat the procedures in Step 1 to Step 6 for $n = 1000$ times and obtain the following quantities.

(a) $EV_i = \frac{\sum_{j=1}^{n} \tilde{a}_{ij}}{n}$

(b) Mean Squared Error, $MSE_i = \frac{\sum_{j=1}^{n} (\tilde{a}_{ij} - \beta_i)^2}{n}$ where $\beta_i; i = 1,2,\ldots,m$ the values of parameters given in Step 1.

(c) Average of variance-covariance matrices computed for different simulated samples, say $V_{\epsilon}$

(d) Reliability functions $\tilde{R}_{ij}(t_i)$ and hazard rate $\tilde{h}_{ij}(t_i); i = 1,2,\ldots,m; j = 1,2,\ldots,n$ evaluate using equations (4.5-4.6) and corresponding $MSEs$ are $\frac{\sum_{i=1}^{n} (\tilde{R}_{ij}(t_i) - R_i(t_i))^2}{n}$ and $\frac{\sum_{i=1}^{n} (\tilde{h}_{ij}(t_i) - h_i(t_i))^2}{n}$ respectively.

Step 8: Obtain Standard Error (SE) of estimates by taking square root of diagonal elements of $V_{\epsilon}$.
### Table 1

Maximum likelihood estimate of parameters, reliability and hazard rates and their efficiency measures. 

\( m = 2, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3 \) and \( t = (0.6536, 0.9849), R(t) = (0.5570, 0.5570), h(t) = (0.8291, 0.7165) \)

<table>
<thead>
<tr>
<th>( u )</th>
<th>( G )</th>
<th>( \hat{\beta}_1 )</th>
<th>( \hat{\beta}_2 )</th>
<th>( \hat{\beta}_3 )</th>
<th>( R_1(t_1) )</th>
<th>( R_2(t_2) )</th>
<th>( h_1(t_1) )</th>
<th>( h_2(t_2) )</th>
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<td>12</td>
<td>EV</td>
<td>1.5366</td>
<td>1.5240</td>
<td>0.5531</td>
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<td>0.0605</td>
<td>0.0080</td>
<td>0.0100</td>
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<td>48</td>
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<td>0.0404</td>
<td>0.0338</td>
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### Table 2

Maximum likelihood estimate of parameters, reliability and hazard rates and their efficiency measures. 

\( m = 3, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4 \), \( t = (0.8536, 0.9849, 0.9146), R(t) = (0.5570, 0.5570, 0.5570), h(t) = (0.8291, 0.7165, 0.7738) \)

<table>
<thead>
<tr>
<th>( u )</th>
<th>( G )</th>
<th>( \hat{\beta}_1 )</th>
<th>( \hat{\beta}_2 )</th>
<th>( \hat{\beta}_3 )</th>
<th>( R_1(t_1) )</th>
<th>( R_2(t_2) )</th>
<th>( R_3(t_3) )</th>
<th>( h_1(t_1) )</th>
<th>( h_2(t_2) )</th>
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</table>
The conclusions for these studies are given below.

We observe that for the known shape parameter \( \alpha \), the means of MLEs for scale parameters \( \beta_i; i = 1,2,\ldots,m \), the reliability characteristics and hazard rates are very close to true value. At average mean square errors are relatively small. Further we observe that the estimates and standard/mean square error are decreasing functions of number \( u \) of each systems put on test. Thus it may concluded that instead usual censoring schemes, present scheme can effectively implemented in simultaneously studying the reliability several types of systems.

### 5.2 Unknown Shape Parameter

Similar study, with not much change in the algorithm, one can make simulation study for the case of unknown shape parameter. We make simulation studies for the set of parameters \( m = 2, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, t = (0.8536, 0.9849), m = 3, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4, t = (0.8536, 0.9849, 0.9146) \) by taking \( n = 1000 \).

### Table 3

<table>
<thead>
<tr>
<th>( u )</th>
<th>( G )</th>
<th>( \hat{\alpha} )</th>
<th>( \hat{\beta}_1 )</th>
<th>( \hat{\beta}_2 )</th>
<th>( \hat{R}_1(t_1) )</th>
<th>( \hat{R}_2(t_2) )</th>
<th>( \hat{h}_1(t_1) )</th>
<th>( \hat{h}_2(t_2) )</th>
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</thead>
<tbody>
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<td>24</td>
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<td>2.9074</td>
<td>1.6450</td>
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<td>36</td>
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<td>48</td>
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<td>0.3843</td>
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<td>-</td>
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<td>60</td>
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<td>1.5886</td>
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<td>0.5501</td>
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<tr>
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<tr>
<td>84</td>
<td>84</td>
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<td>0.3062</td>
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<tr>
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<td>0.5585</td>
<td>0.5540</td>
<td>0.8717</td>
<td>0.7701</td>
</tr>
</tbody>
</table>

**International Journal of Statistics and Reliability Engineering**
In the presence of unknown shape parameter $\alpha$, from Table 3 and Table 4, it is seen that the MLEs of scale parameters $\beta_i; i = 1, 2, \ldots, m$, the reliability characteristics and hazard rate are reaching close to their true values. However the convergence rate is slow compared to the convergence rate when shape parameter $\alpha$ is known. Perhaps, it may be the effect of estimate of unknown shape parameter $\alpha$. Further we can say, somewhat large sample size is required than what we consider for the estimates to reach their true values.

6. Testing of Hypotheses

The proposed design will have significance only when we are able to ascertain that the $m$ type of systems are not all have identical life time. This can be done by developing ANOVA approach for the proposed design. However, we will utilize likelihood approach to develop a test. The testing hypothesis problem is to test

$$H_0: \beta_1 = \beta_2 = \cdots = \beta_m = \beta \quad \text{against} \quad H_1: \beta_i \neq \beta_j \text{ for at least one pair } (i,j), i \neq j. \quad (6.1)$$

As we are considering maximum likelihood estimation, the use likelihood ratio test is much convenient. The test statistic is

$$\lambda_{LR} = \frac{\max_{\alpha, \beta} L(t, \beta, \alpha)}{\max_{\alpha, \beta} L(t, \tilde{\beta}, \alpha)}$$
The test based on \(-2 \ln (\hat{\lambda}_{LR})\) rejects $H_0$ in support of $H_1$ if it is larger than upper $\alpha$th cut point of chi-square distribution $(m-1)$ degrees of freedom.

### 6.1 Computation of Likelihood under $H_0$

The log likelihood $\ln L_G$ under null hypothesis from equation (4.1) we have,

\[
\ln L_G = \ln(\text{const}) + G(\ln \alpha + \ln \beta) - \beta \sum_{g=1}^{G} t_g + (\alpha - 1) \sum_{g=1}^{G} \ln(1 - e^{-\beta t_g})
\]

\[
+ (m - 1) \sum_{g=1}^{G} \ln \left[ 1 - (1 - e^{-\beta t_g})^\alpha \right] + m(u - G) \ln \left[ 1 - (1 - e^{-\beta t_G})^\alpha \right] \tag{6.2}
\]

Differentiate (6.2) with respect to $\alpha$ and $\beta$ we get,

\[
\frac{\partial \ln L_G}{\partial \alpha} = \frac{G}{\alpha} + \sum_{g=1}^{G} \ln(1 - e^{-\beta t_g}) - (m - 1) \sum_{g=1}^{G} \frac{(1 - e^{-\beta t_g})^\alpha \ln(1 - e^{-\beta t_g})}{[1 - (1 - e^{-\beta t_G})^\alpha]}
\]

\[
- m(u - G) \frac{(1 - e^{-\beta t_G})^\alpha \ln(1 - e^{-\beta t_G})}{[1 - (1 - e^{-\beta t_G})^\alpha]} \tag{6.3}
\]

\[
\frac{\partial \ln L_G}{\partial \beta} = G - \sum_{g=1}^{G} t_g + (\alpha - 1) \sum_{g=1}^{G} \frac{e^{-\beta t_g}}{1 - e^{-\beta t_g}}
\]

\[
- (m - 1) \alpha \sum_{g=1}^{G} \frac{(1 - e^{-\beta t_g})^{\alpha - 1} \ln(1 - e^{-\beta t_g})}{[1 - (1 - e^{-\beta t_G})^\alpha]}
\]

\[
- m \alpha (u - G) t_G \frac{(1 - e^{-\beta t_G})^{\alpha - 1} \ln(1 - e^{-\beta t_G})}{[1 - (1 - e^{-\beta t_G})^\alpha]} \tag{6.4}
\]

Differentiate (6.3) and (6.4) with respect to $(\alpha, \beta)$ and $\beta$ respectively we have

\[
\frac{\partial^2 \ln L_G}{\partial \alpha^2} = - \frac{G}{\alpha^2} - (m - 1) \sum_{g=1}^{G} \frac{(1 - e^{-\beta t_g})^\alpha [\ln(1 - e^{-\beta t_g})]^2}{[1 - (1 - e^{-\beta t_G})^\alpha]^2}
\]

\[
- m(u - G) \frac{(1 - e^{-\beta t_G})^\alpha [\ln(1 - e^{-\beta t_G})]^2}{[1 - (1 - e^{-\beta t_G})^\alpha]^2} \tag{6.5}
\]

\[
\frac{\partial^2 \ln L_G}{\partial \alpha \partial \beta} = \sum_{g=1}^{G} t_g \left[ \frac{e^{-\beta t_g}}{1 - e^{-\beta t_g}} \right]
\]

\[
- (m - 1) \alpha \sum_{g=1}^{G} t_g \left[ \frac{e^{-\beta t_g} (1 - e^{-\beta t_G})^{\alpha - 1}}{[1 - (1 - e^{-\beta t_G})^\alpha]^2} \right] \left[ \frac{1 - (1 - e^{-\beta t_G})^\alpha}{\alpha} + \ln(1 - e^{-\beta t_G}) \right]
\]

\[
- m \alpha (u - G) t_G \left[ \frac{e^{-\beta t_G} (1 - e^{-\beta t_G})^{\alpha - 1}}{[1 - (1 - e^{-\beta t_G})^\alpha]^2} \right] \left[ \frac{1 - (1 - e^{-\beta t_G})^\alpha}{\alpha} + \ln(1 - e^{-\beta t_G}) \right] \tag{6.6}
\]
The likelihood equation (6.4) is not mathematically tractable for known as well as unknown shape parameter we use the Newton-Raphson method to obtain the estimate of parameter $\beta$. Here we deal with only known shape parameter.

We demonstrate the test procedure for $m=2$. We generate data under our design for the parameter values under $H_1: \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1$. Then carryout the test procedure as suggested above. The procedure is repeated for the different choices of $u$ and $G$. The results are produced in the table.

<table>
<thead>
<tr>
<th>$u$</th>
<th>$G$</th>
<th>$\beta$</th>
<th>$\hat{\beta}_1$</th>
<th>$\hat{\beta}_2$</th>
<th>$LL_{H_1}$</th>
<th>$LL_{H_2}$</th>
<th>$\chi^2$</th>
<th>p-value</th>
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<td>48</td>
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</table>

From the above table we infer that the power the test is poor for small sizes. However the as sample size becomes 48(total number failures observed irrespective type of systems) it exhibits its power in identifying the alternative. The problem of obtaining a test for small sizes is still open research. The consistency of the test is also inferred as sample size tends to 96 the p value becomes almost zero up to two digits.

7. Design Optimality Criteria

In this Section, three optimality criteria namely; “A-optimality”, “D-optimality” and “E-optimality” for SRD21GED are discussed. For details one may refer Shah, K. R. et al. (1989). These optimality criteria respectively defined as

**A – Optimality Criterion:** The design is said to be an A-optimal if it minimizes the trace of inverse of information matrix or trace of estimate of variance-covariance matrix of estimators’ i.e. $tr(\hat{V}^{-1})$. 

$$
\frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{G}{\beta^2} - (m-1)\alpha \sum_{g=1}^{G} \left[ \frac{g^2 e^{-\beta g} (1 - e^{-\beta g})^{g-2}}{[1 - (1 - e^{-\beta g})^g]^2} \right] \left[ \alpha e^{-\beta g} - 1 + (1 - e^{-\beta g})^g \right]
$$

$$
-m\alpha (u - G) \left[ \frac{\alpha e^{-\beta g} - 1 + (1 - e^{-\beta g})^g}{[1 - (1 - e^{-\beta g})^g]^2} \right]
$$

(6.7)
**D – Optimality Criterion:** The design is said to be D-optimal if it minimizes the determinant of inverse of information matrix or determinant of estimate of variance-covariance matrix of estimators’ i.e. $\text{tr}(\hat{V}^{-1})$.

**E – Optimality Criterion:** The design is said to be E-optimal if it maximizes the minimum eigen-values of the information matrix ($\hat{V}$).

The variance-covariance matrix of estimates are not mathematically tractable we simulate it using Monte-Carlo technique for both cases when shape parameter is known and when it is unknown. Further we obtain optimality criterion (A, D and E) for the number of units $u$ changes from 12 to 96. The simulated results are given in Table 6 and Table 7 for known shape parameter and in Table 8 and Table 9 for unknown shape parameter.

**Table 6**

Optimality Criteria for Self Relocated Design and Generalized Type II Censoring Design for Known Shape Parameter

<table>
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<tr>
<th>$u$</th>
<th>$G$</th>
<th>$A$</th>
<th>$D$</th>
<th>$E$</th>
<th>$A$</th>
<th>$D$</th>
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**Table 7**

Optimality Criteria for Self Relocated Design and Generalized Type II Censoring Design for Known Shape Parameter

<table>
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</tr>
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<td>0.00001</td>
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<td>96</td>
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<td>0.00000</td>
<td>45.371</td>
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<td>0.00001</td>
<td>44.440</td>
</tr>
</tbody>
</table>
Table 8  
Optimality Criteria for Self Relocated Design and Generalized Type II Censoring Design for  
Unknown Shape Parameter  
\( m = 2, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, n = 1006, \tau = (0.8536, 0.9849), R(t) = (0.5570, 0.5570), h(t) = (0.8291, 0.7185) \)

<table>
<thead>
<tr>
<th>( u )</th>
<th>( G )</th>
<th>( A )</th>
<th>( D )</th>
<th>( E )</th>
<th>( A )</th>
<th>( D )</th>
<th>( E )</th>
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<tbody>
<tr>
<td>24</td>
<td>24</td>
<td>0.3144</td>
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<td>4.060</td>
<td>0.6545</td>
<td>0.10511</td>
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<td>0.2790</td>
<td>0.01907</td>
<td>6.280</td>
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<tr>
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<td>0.00230</td>
<td>11.291</td>
<td>0.2142</td>
<td>0.01123</td>
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<tr>
<td>72</td>
<td>72</td>
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<td>13.858</td>
<td>0.1746</td>
<td>0.00748</td>
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<tr>
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<td>18.442</td>
<td>0.1276</td>
<td>0.00400</td>
<td>13.790</td>
</tr>
</tbody>
</table>

Table 9  
Optimality Criteria for Self Relocated Design and Generalized Type II Censoring Design for  
Unknown Shape Parameter  
\( m = 3, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4, n = 1006, \tau = (0.8536, 0.9849, 0.9146), R(t) = (0.5570, 0.5570, 0.5570), h(t) = (0.8291, 0.7185, 0.7738) \)

<table>
<thead>
<tr>
<th>( u )</th>
<th>( G )</th>
<th>( A )</th>
<th>( D )</th>
<th>( E )</th>
<th>( A )</th>
<th>( D )</th>
<th>( E )</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
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<td>0.5716</td>
<td>0.00320</td>
<td>2.556</td>
<td>2.1530</td>
<td>0.36212</td>
<td>1.210</td>
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<tr>
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<td>36</td>
<td>0.3697</td>
<td>0.00090</td>
<td>5.536</td>
<td>1.940</td>
<td>0.06192</td>
<td>2.220</td>
</tr>
<tr>
<td>48</td>
<td>48</td>
<td>0.2669</td>
<td>0.00030</td>
<td>5.930</td>
<td>0.8149</td>
<td>0.01967</td>
<td>3.260</td>
</tr>
<tr>
<td>60</td>
<td>60</td>
<td>0.2134</td>
<td>0.00020</td>
<td>6.310</td>
<td>0.6131</td>
<td>0.00836</td>
<td>4.300</td>
</tr>
<tr>
<td>72</td>
<td>72</td>
<td>0.1779</td>
<td>0.00009</td>
<td>8.310</td>
<td>0.6242</td>
<td>0.00881</td>
<td>4.170</td>
</tr>
<tr>
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<td>0.1497</td>
<td>0.00005</td>
<td>9.894</td>
<td>0.4156</td>
<td>0.00260</td>
<td>6.290</td>
</tr>
<tr>
<td>96</td>
<td>96</td>
<td>0.1381</td>
<td>0.00004</td>
<td>11.236</td>
<td>0.3615</td>
<td>0.00171</td>
<td>7.180</td>
</tr>
</tbody>
</table>

From the Table 6 to Table 9 we have the following comparison of SRD and generalized Type II censoring.

i. The performance SRD and generalized Type II censoring designs are same with respect to A-optimality and D-optimality criteria when shape parameter is assumed to be known.

ii. In the case of unknown shape parameter SRD is both A-optimal and D-optimal.

iii. SRD is always E-optimal as compared to generalized Type II censoring.

8. The Cost Function

Here, we reproduce the cost function in Section 8.1 and Section 8.2, suggested by Srivastava (1987) with some changes in notation for simulation study purpose.

8.1 The Cost Function under SRD

Suppose we start with \( u \) machines of each of the \( m \) types. Let \( C_1 \) be the cost involved with the failure of a machine irrespective of types. For some experiments, any ‘failed’ machine may be rebuilt sufficiently well; for others, failure may mean the machine is total loss. If, in the
whole experiment, the failure of a total of \( G \) machines is allowed, the total cost involved on this account is \( C_1G \); notice that this cost is fixed and pre-planned.

Next, let \( C_2 \) be the cost involved for keeping any machine under test for one unit of time. If under the SRD21, the failure times are \( (t_1, t_2, \ldots, t_G) \), then the “total machine test time”, i.e. the total number of units of time all the \( um \) machines taken together are under test, is \( t_1^{SRD} \), where
\[
t_1^{SRD} = m(u-1)(t_2 - t_1) + m(u-2)(t_3 - t_2) + \ldots + m(u-G+1)(t_G - t_{G-1}) \\
= m[t_1 + t_2 + \ldots + t_{G-1} + (u - G + 1)t_G]
\]
(8.1)
Thus, the total cost involved in keeping the machines under test is \( C_2t_1^{SRD} \).

Another component of the total cost comes from the total length of time the whole experiment is continued. It is clear from the scheme that the experiment lasts for \( t_G \) units of time. If \( C_3 \) is the overhead cost incurred for each unit of time the experiment continues, then the cost from this source is \( C_3t_G \).

The fourth component of the cost relates to the total number of machines used in the whole experiment, which under SRD21 equal to \( mu = N \). This cost deals with the actual procurement, storage and handling of the machines. We shall assume that \( \gamma(N) \), an increasing function of \( N \), is the cost of putting \( N \) machines on experiment. In some cases, \( \gamma(N) \) may increase slower than \( N \) or \( \gamma \) may even be discontinuous.

Finally, let \( C_0 \) denote the overhead cost associated with conducting the experiment. We shall assume that \( C_0 \) represents general costs such as those associated with planning the experiment, administrative and consultancy costs, etc. However, we make the assumption that \( C_0 \) is independent of \( N \), the total number of machines used in the experiment; such costs are already included in \( \gamma(N) \). With the above assumptions, the actual cost of the experiment under SRD21 is \( C_a^{SRD} \) where
\[
C_a^{SRD} = C_0 + C_1G + C_2t_1^{SRD} + C_3t_G + \gamma(N)
\]
(8.2)

8.2 The Cost Function under Generalized Type-II Censoring

Suppose we start life testing experiment with \( u \) machines each from \( m \) makes or brands. Further, we shall assume that cost of failure of a system is constant, say \( C_1 \), irrespective of the brand. Since the experiment is terminated after observing total \( G = m \ G^* \) where, \( G^* \) is a fixed number failure being observed on each brand of systems then the total cost due to failure is \( G = m \ C_1 G^* \) which is fixed and pre-planned. In this scheme there are \( m \) separate sub experiments in the sense that the way we observe the machines of type \( i \) does not
depend upon the failures of machines of type \( i' \), for \( i \neq i' \). Let \( t_{G^*i}; i = 1,2, \ldots, m \) denote the time of failure of \( G^* \)th machine of the \( i \)th type. Define \( t_{\text{max}} \) by

\[
t_{\text{max}} = \max(t_{G^*1}, t_{G^*2}, \ldots, t_{G^*m}).
\]  

(8.3)

So that \( t_{\text{max}} \), denotes the duration of experiment for which the whole experiment under Type II censoring lasts. Under this experimental scheme, the component of cost from the view point of length of the experiment, is \( C_3 t_{\text{max}} \).

Now we shall consider the cost related with total experimental time for all the units put under test. Suppose \( C_2 \) is the cost per unit time associated with testing time of a machine, irrespective of the brand. Let \( t_{1i}, t_{2i}, \ldots, t_{G^*i} \) be the failure times of \( G^* \) systems out of \( u \) systems of type \( i, i = 1,2, \ldots, m \). Then the total time the on test for the \( i \)th sub experiment is

\[
u t_{1i} + (u - 1)(t_{2i} - t_{1i}) + (u - 2)(t_{3i} - t_{2i}) + \cdots + (u - G^* + 1)(t_{G^*i} - t_{G^*-1i}).
\]

Hence, the total machine time under Type II censoring is \( t_2 \) where

\[
t_2 = \sum_{i=1}^{G^*} t_{1i} + t_{2i} + \cdots + (u - G^* + 1)t_{G^*i}.
\]  

(8.4)

The contribution of this cost is \( t_2 C_2 \).

The fourth component of the cost relates to the total number of machines used in the whole experiment, which under generalized Type II censoring equal to \( m u = N \). This cost deals with the actual procurement, storage and handling of the machines. We shall assume that \( \gamma(N) \), an increasing functions of \( N \), is the cost of putting \( N \) machines on experiment. In some cases, \( \gamma(N) \) may increase slower than \( N \) or \( \gamma \) may even be discontinuous.

Finally, let \( C_0 \) denote the overhead cost associated with conducting the experiment. We shall assume that \( C_0 \) represents general costs such as those associated with planning the experiment, administrative and consultancy costs, etc. However, we make the assumption that \( C_0 \) is independent of \( N \), the total number of machines used in the experiment; such costs are already included in \( \gamma(N) \). With the above assumptions, the actual cost of the experiment under generalized Type II censoring design is

\[
C^H_2 = C_0 + C_1 G + C_2 t_2 + C_3 t_{\text{max}} + \gamma(N)
\]  

(8.5)

We carry out Monte-Carlo simulation study for the choice of cost \( C_0 = 100, C_1 = 5, C_2 = 10, C_3 = 10 \) and \( \gamma(N) = 0.5N \) for the cost effectiveness of Self Relocated Designs and Type II censoring scheme. We consider \( m = 2 \) and \( m = 3 \) and results are given in Table 10 and Table 11 respectively. Further, we fix total number of failures \( G \) and change the number of units to be tested in the experiment. The results are given in Table 12.
### Table 10
Comparative Table of Total time under experiment, duration of experiment and cost of experiment

\[ m = 2, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3 \]

<table>
<thead>
<tr>
<th>( u )</th>
<th>( G )</th>
<th>( t_{1}^{SRD} )</th>
<th>( t_G )</th>
<th>( C_{\alpha}^{SRD} )</th>
<th>( t_2 )</th>
<th>( t_{\text{max}} )</th>
<th>( C_{\alpha}^{II} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>12</td>
<td>18.7820</td>
<td>1.7329</td>
<td>377.149</td>
<td>19.3146</td>
<td>1.0437</td>
<td>375.583</td>
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<tr>
<td>24</td>
<td>24</td>
<td>37.0226</td>
<td>1.9533</td>
<td>633.759</td>
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<td>1.0651</td>
<td>641.652</td>
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<tr>
<td>36</td>
<td>36</td>
<td>55.4445</td>
<td>2.1190</td>
<td>891.635</td>
<td>58.2674</td>
<td>1.0770</td>
<td>909.444</td>
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<tr>
<td>48</td>
<td>48</td>
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<td>1147.635</td>
<td>78.0618</td>
<td>1.0853</td>
<td>1179.471</td>
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<tr>
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<td>92.7800</td>
<td>2.2682</td>
<td>1410.482</td>
<td>97.3652</td>
<td>1.0834</td>
<td>1444.486</td>
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<td>1661.905</td>
<td>116.3868</td>
<td>1.0770</td>
<td>1706.684</td>
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<tr>
<td>84</td>
<td>84</td>
<td>129.0872</td>
<td>2.4266</td>
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<td>135.6755</td>
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<tr>
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<td>155.1307</td>
<td>1.0789</td>
<td>2238.096</td>
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</tbody>
</table>

### Table 11
Comparative Table of total time under experiment, duration of experiment and cost of experiment

\[ m = 3, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4 \]

<table>
<thead>
<tr>
<th>( u )</th>
<th>( G )</th>
<th>( t_{1}^{SRD} )</th>
<th>( t_G )</th>
<th>( C_{\alpha}^{SRD} )</th>
<th>( t_2 )</th>
<th>( t_{\text{max}} )</th>
<th>( C_{\alpha}^{II} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>12</td>
<td>21.8509</td>
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<td>409.363</td>
<td>23.1533</td>
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<td>1350.869</td>
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<td>72</td>
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<td>84</td>
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<td>0.7908</td>
<td>2287.766</td>
</tr>
<tr>
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<td>96</td>
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<td>2483.437</td>
<td>186.4134</td>
<td>0.7911</td>
<td>2596.045</td>
</tr>
</tbody>
</table>

### Table 12
Comparative Table of total time under experiment, duration of experiment and cost of experiment for fixed failures

\[ m = 3, G = 12, \alpha = 2.5, \beta_1 = 1.5, \beta_2 = 1.3, \beta_3 = 1.4 \]

<table>
<thead>
<tr>
<th>( u )</th>
<th>( t_{1}^{SRD} )</th>
<th>( t_G )</th>
<th>( C_{\alpha}^{SRD} )</th>
<th>( t_2 )</th>
<th>( t_{\text{max}} )</th>
<th>( C_{\alpha}^{II} )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>43.6102</td>
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<tr>
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<td>654.734</td>
</tr>
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<td>0.6000</td>
<td>641.021</td>
</tr>
<tr>
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<td>32.7037</td>
<td>0.5419</td>
<td>615.456</td>
</tr>
<tr>
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<td>0.5625</td>
<td>548.649</td>
<td>31.7662</td>
<td>0.5248</td>
<td>610.410</td>
</tr>
<tr>
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<td>542.145</td>
<td>30.7383</td>
<td>0.5084</td>
<td>604.467</td>
</tr>
</tbody>
</table>
From the Table 10 to Table 12 we observe that SRD design is more cost effective as compared to generalized Type II censoring design.

9. Concluding Remarks

In this article we study SRD proposed by Srivastava [1987] and use generalized exponential distribution as lifetime model proposed by Gupta and Kundu [1999] and suggest the maximum likelihood estimation procedure to estimate parameters. Further, likelihood ratio test for homogeneity of different types of systems is carried out and through simulation we show that the said procedure is good in terms of convergence to their true values. The SRD design is always E-optimal and is A and D-optimal when shape parameter is unknown. Further, the SRD is cost effective as compared to generalized Type II censoring. Similar kind of work can be carried out for generalized exponential family of distributions. The work in this direction is in progress.

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References


